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ON THE DIFFUSION OF A CONDUCTING FLUID ACROSS A MAGNETIC FIELD

by

Marian H. Rose

July 1, 1956

institute of mathematical sciences

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Abstract

The purpose of this report is to examine the concepts upon which diffusion theory is based and, in particular, to show that the rate of diffusion of a conducting fluid across a magnetic field is independent of the field if the proper scaling is used.

Introduction

The problem of the diffusion of an ionized fluid across a magnetic field has been investigated both theoretically and experimentally(1). The aim has been to determine how the coefficient of conduction or diffusion is affected by the presence of a magnetic field B. Both theory and measurements have shown that the rate of diffusion perpendicular to B is decreased, this being attributed to a decrease in the coefficient of conductivity. The latter is usually defined as the coefficient of proportionality between the current J and the electric field E. This, as we shall show later, is an ambiguous statement and the dependence of conductivity on B depends on how exactly conductivity is defined. The question as to which inverse power of B the conductivity depends on experimentally has been settled only recently by Simon in favor of a $\frac{1}{B^2}$ law(2).

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(1) Chapman and Cowling. The Mathematical Theory of non-uniform gases. Cambridge University Press, 1939

A. Guthrie and R. K. Wakerling. The Characteristics of Electrical Discharges in Magnetic Fields. (McGraw-Hill Book Co., Inc., New York, 1949) National Nuclear Energy Series, Plutonium Project Record, Vol. 5 Division 1.

A. Simon. Ambipolar Diffusion in a Magnetic Field. Physical Review, Vol. 98, 317-318, 1955.

In this paper, the problem will be treated from the view point of continuum theory, most existing theories of diffusion having been based on kinetic theory. The purpose is to elucidate some of the underlying concepts rather than to present explicit formulas for diffusion rates. These will be given in a note to follow since they might be of some use in experimental work. In particular, we shall show that when the equations of the fluid are expressed in dimensionless form, only one essential parameter β remains. If a strict comparison is made between experiments, this parameter should be kept constant. We shall show that as a consequence, the velocity of diffusion is independent of B. If, however, β is allowed to vary, i.e., strict scaling is no longer maintained, then a dependence on B can be obtained. For one special type of scaling, the velocity of diffusion will be shown to be proportional to $\frac{1}{B^2}$ for large B, in agreement with experiment.

The conclusion that the coefficient of conductivity is independent of B is, at first sight, rather surprising. However, it does not contradict any previously established theories or experimental results. Rather, it serves to emphasize that this coefficient has no precise meaning, independent of the definition which is assigned to it or the experiment which measures it. The same situation exists in regard to the notion of "mean free path" which may have many different meanings depending on how it is measured. In order to illustrate this point: Ohm's law may be written as $J' = \gamma E' + \delta E' x B$ where γ and δ depend on B, J' and E' being the current and electric field as viewed from a coordinate system moving with the velocity of the fluid. If γ is identified with the coefficient of conductivity, we may state that the latter depends on B. However, if the formula is inverted with E' expressed in terms of J', then the coefficient of J' turns out to be independent of B. Another example which could be used arises from the equation of motion for B, namely,

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(2) Cf. reference (1).

$$\frac{\partial B}{\partial t} + (u \cdot \nabla) B + B(\nabla \cdot u) = \frac{1}{\mu\sigma} \nabla^2 B$$

where u is the velocity of the fluid, μ, the permeability, and σ, the conductivity in the absence of B. Apart from the convection terms, the equation has the form of a diffusion equation for B - hence, we state that in this case, the rate of diffusion is independent of B. Since these points have been treated extensively by Grad(3), we shall not elaborate on them further. Instead, the author takes pleasure in thanking Professor Grad for many useful discussions on these topics.

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(3) Harold Grad, Notes on Magneto-Hydrodynamics, NYO-6486, Nos. IV and VII.

Basic assumptions and equations. We consider a steady-state flow in which all quantities depend on x alone, x being the direction of flow perpendicular to B.

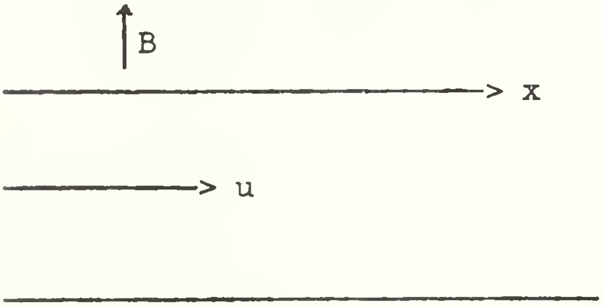

Figure 1.

We assume that the following conditions prevail:

1) Departures from neutrality are so small that the electrical forces acting on a volume element of the fluid may be neglected as compared to the magnetic forces.

2) The flow is isothermal, hence, $p = \alpha\rho$ where p is the pressure, ρ, the density and α, a parameter. A more realistic assumption would be that energy flow takes place. However, this would not change the qualitative nature of the results although quantitatively they would differ. Isothermal flow may even be a good approximation in some cases, e.g., when radiation effects are large.

The equations describing the motion of the fluid are the conservation equations of mass and momentum together with Maxwell's equations and Ohm's law.

Conservation of mass: $\rho u = M$ or $pu = \alpha M$ (1)
where M is a constant
of integration.

Conservation of momentum: $\rho u \cdot \nabla u + \text{grad } p = J \times B$ (2)

Maxwell's equations: $\text{curl } E = 0$ (3)

$\text{curl } B = \mu J$ (4)

Ohm's law: $E' = RJ + \xi J \times B$ (5)

where $E' = E + u \times B$

Isothermal flow: $p = \alpha\rho$ (6)

We note that $R = \frac{1}{\sigma}$, hence, is independent of B whereas ξ may depend on B. The second term on the right-hand side of equation (5) is the Hall vector term. We shall now show that, for the particular geometry considered here, this term drops out of the equation of motion for B. The latter is obtained from equations (3), (4) and (5). From curl $E = 0$ we obtain $-\text{curl}(u \times B) + R \text{ curl } J + \xi \text{ curl } (J \times B) = 0$ (7)
But $\text{curl}(J \times B) = 0$ since J only has a y component. Using equations (4) and (7) and div $B = 0$, we obtain

$$\text{curl curl } B = \mu \text{ curl } J = \frac{\mu}{R} \text{ curl } (u \times B) \quad (8)$$

This is exactly the relation obtained when Ohm's law is written in the form $J = \sigma(E + u \times B)$, without the Hall vector.

Equation (8) may be integrated, yielding

$$Bu = \frac{1}{\mu\sigma}\frac{dB}{dx} + \varepsilon \qquad (9)$$

where ε is a constant of integration equal to the y component of the electric field. By using equation (4), the momentum equation may also be expressed in integrated form:

$$Mu + p + \frac{B^2}{2\mu} = \pi \qquad (10)$$

where π is another integration constant. Equations (1), (9) and (10) constitute the basic system for determining the quantities u, p and B. It can be used to describe either shock flows or diffusive phenomena. The difference between the two cases shows up in the number of parameters that are required in order to specify the problem. This may be understood more easily by using a cylindrical geometry in which the magnetic field has the direction of the axis and the flow lies in planes perpendicular to the axis. All variables depend on r alone, namely, the distance along the radius from the axis of the cylinder. The equation $\frac{\partial B}{\partial t} = -\,\mathrm{curl}\ E$, written in integral form is

$$\int_A \frac{\partial B}{\partial t} \cdot dS = -\int_A \mathrm{curl}\ E \cdot dS = -\oint E \cdot d\ell$$

where $d\ell$ is an element of length along the curve enclosing the area A, which is perpendicular to and contains the axis of the cylinder. If no sources are enclosed, then the expression on the right vanishes or $\frac{\text{curl } B}{\mu\sigma}$ + + uxB = grad φ. Now, grad φ lies in the θ direction and depends on r alone. In order to satisfy these requirements, φ must be directly proportional to θ. However, the integral would then no longer vanish. The only way to satisfy all requirements is for φ to vanish everywhere hence for grad φ to vanish. In the case of shock flows, sources are needed along the axis of the cylinder in order to maintain a steady flow of fluid through the shock front, hence grad $\varphi \equiv \varepsilon \neq 0$. In the case of diffusion, however, this is not necessary, hence grad $\varphi \equiv \varepsilon = 0$. For shocks, therefore, an extra parameter (hence boundary condition) is needed in order to specify the problem. This case will be treated in a paper to follow and will not be pursued here. The same argument can be extended somewhat artificially to the plane case by allowing r --> ∞ and the sources to recede to infinity.

Returning to diffusion, it would appear that three parameters are necessary in the problem, namely, M, π and α. However, if equations (1), (9) and (10) are expressed in dimensionless form, only one essential parameter β remains. Introducing the dimensionless

quantities $\bar{p} = \frac{p}{\pi}$, $\bar{B} = \frac{B}{\sqrt{\mu\pi}}$, $\bar{u} = \frac{u}{\sqrt{a}}$, $\bar{x} = \sqrt{a}\,\mu\sigma x$, $\bar{\beta} = \frac{\sqrt{aM}}{\pi}$, these equations become

$$\bar{p}\bar{u} = \bar{\beta} \tag{11}$$

$$\bar{\beta}\bar{u} + \bar{p} + \frac{\bar{B}^2}{2} = 1 \tag{12}$$

$$\bar{B}\bar{u} = \frac{d\bar{B}}{d\bar{x}} \tag{13}$$

Conclusions

It is now a simple matter to prove the assertions made in the introduction.

1) The only strict scaling of a flow is one in which $\bar{\beta}$ is maintained constant. In such a case we might wish to compare experiments having large and small magnetic fields. For large fields, π would have to be increased in order to keep $\bar{B}$ constant hence p would also have to be increased in order to keep $\bar{p}$ constant. However u would be unaffected. We conclude that u is independent of B in this strict sense.

2) Other types of scaling are possible. These, however, are only local and allow the dependence on x to change. For example $\bar{\beta}$ and π might be allowed to vary keeping $\frac{dp}{dx}$ constant. It is then easily shown that $u \propto \frac{1}{B^2}$ for large B. This agrees exactly with Chapman and Cowling's result(3) which is derived on the basis of kinetic theory using the same type of scaling.

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(3) Chapman and Cowling. Ibid, pp. 327-328.

3) The basic system of equations yields

$$\bar{u} = \frac{-\left(1 - \frac{\bar{B}^2}{2}\right) \pm \sqrt{\left(1 - \frac{\bar{B}^2}{2}\right)^2 - 4\bar{\beta}^2}}{\bar{\beta}}$$

Hence, for a given flow, u is in general not proportional to any specific power of B.

4) ρu, the mass flow, is independent of B since B depends on x and ρu does not.

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